5) זריקה אנכית כמודל דינמי: גוף בעל מסה m נזרק אנכית כלפי מעלה במהירות התחלתית v_0 בתווך בעל התנגדות $k\cdot v$ (k קבוע). הזנח שינויים בכח המשיכה בגלל שינויי גובה. מצא את הגובה המקסימלי אליו יגיע? כמה זמן דרוש לו להגיע לגובה זה? כיצד משתנה התוצאה כאשר $k\to 0$?

$$\sum F = ma \quad => \quad -(kv+mg) = m\frac{dv}{dt} \quad => \quad -dt = m\frac{dv}{kv+mg} \quad =>$$

$$=> \quad -\frac{k}{m}dt = \frac{dv}{v+\frac{mg}{k}} \quad => \quad C-\frac{k}{m}\int dt = \int\frac{dv}{v+\frac{mg}{k}} \quad =>$$

$$C-\frac{k}{m}t = \ln\left(v+\frac{mg}{k}\right) \quad => \quad v+\frac{mg}{k} = e^{\ln C-\frac{k}{m}t} \quad => \quad v_{(t)} = Ce^{-\frac{k}{m}t}-\frac{mg}{k}$$

$$v_{(0)} = v_0 \quad => \quad v_0 = Ce^0-\frac{mg}{k} \quad => \quad v_0+\frac{mg}{k} = C$$

$$v_{(t)} = \left(v_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}-\frac{mg}{k} \qquad at\ maximum\ hight\ v_{(t)} = 0\ \ therefore:$$

$$0 = (v_0k+mg)e^{-\frac{k}{m}t}-mg \quad => \quad e^{-\frac{k}{m}t} = \frac{mg}{v_0k+mg} \quad =>$$

$$-\frac{k}{m}t = \ln\frac{mg}{v_0k+mg} \quad => \quad t_{\max hight} = \frac{m}{k}\ln\frac{v_0k+mg}{mg}$$

$$h_{(t)} = \int v_{(t)}dt = \int\left[\left(v_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}-\frac{mg}{k}\right]dt = -\frac{m}{k}\left(v_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}-\frac{mg}{k}t+C$$

$$h_{(0)} = 0 \quad => \quad 0 = -\frac{m}{k}\left(v_0+\frac{mg}{k}\right)+C \quad => \quad \frac{m}{k}\left(v_0+\frac{mg}{k}\right) = C$$

$$h_{(t)} = -\frac{m}{k}\left(v_0+\frac{mg}{k}\right)e^{-\frac{k}{m}t}-\frac{mg}{k}t+\frac{m}{k}\left(v_0+\frac{mg}{k}\right)$$

$$h_{(t)} = \frac{m}{k}\left(v_0+\frac{mg}{k}\right)\left(1-e^{-\frac{k}{m}t}\right)-\frac{mg}{k}t$$

$$h_{max} = h_{\left(t_{\max hight}\right)} = \frac{m}{k}\left(v_0+\frac{mg}{k}\right)\left(1-e^{-\frac{k}{m}\cdot\frac{m}{k}\ln\frac{v_0k+mg}{mg}}\right)-\frac{mg}{k}\cdot\frac{m}{k}\ln\frac{v_0k+mg}{mg}$$

$$h_{max} = \frac{m}{k}\left[\left(v_0+\frac{mg}{k}\right)\left(1-e^{-\ln\frac{v_0k+mg}{mg}}\right)-\frac{mg}{k}\ln\frac{v_0k+mg}{mg}\right]$$

$$h_{max} = \frac{m}{k}\left[\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{\ln\frac{mg}{v_0 k + mg}}\right) - \frac{mg}{k}\ln\frac{v_0 k + mg}{mg}\right]$$

$$h_{max} = \frac{m}{k}\left[\left(v_0 + \frac{mg}{k}\right)\left(1 - \frac{mg}{v_0 k + mg}\right) - \frac{mg}{k}\ln\frac{v_0 k + mg}{mg}\right]$$

$$h_{max} = \frac{m}{k}\left[\frac{v_0 k + mg}{k}\frac{v_0 k}{v_0 k + mg} - \frac{mg}{k}\ln\frac{v_0 k + mg}{mg}\right]$$

$$h_{max} = \frac{m}{k}\left[v_0 - \frac{mg}{k}\ln\left(\frac{v_0 k + mg}{mg}\right)\right]$$

What happens when $K \to 0$?

$$h_{max} = m \cdot \frac{v_0 k - mg\ln\frac{v_0 k + mg}{mg}}{k^2} \qquad \textit{reordering for Lopital}$$

$$\lim_{k\to 0} h_{max} = m \cdot \frac{0 - mg \cdot 0}{0} = \left[\frac{0}{0}\right] \quad => \quad \textit{Lopital}$$

$$\lim_{k\to 0} h_{max} = m\frac{v_0 - mg\frac{mg}{v_0 k + mg}\cdot\frac{v_0}{mg}}{2k} = m\frac{v_0 - \frac{mgv_0}{v_0 k + mg}}{2k} = m\frac{{v_0}^2 k}{2(v_0 k + mg)k} =$$

$$= \frac{m{v_0}^2}{2(v_0 k + mg)} = \frac{m{v_0}^2}{2(v_0 \cdot 0 + mg)} = \frac{{v_0}^2}{2g}$$

$\frac{{v_0}^2}{2g}$ is indeed the formula for h_{max} when motion in vacuum (no resistance) is assumed.

Another approach is to substitute $\varepsilon - \frac{\varepsilon^2}{2}$ for $\ln(1+\varepsilon)$ when $\varepsilon \to 0$:

$$h_{max} = \frac{m}{k}\left[v_0 - \frac{mg}{k}\ln\left(1 + \frac{v_0 k}{mg}\right)\right]$$

$$\lim_{k\to 0} h_{max} = \frac{m}{k}\left[v_0 - \frac{mg}{k}\left(\frac{v_0 k}{mg} - \frac{{v_0}^2 k^2}{2m^2 g^2}\right)\right] = \frac{m}{k}\left[v_0 - v_0 + \frac{{v_0}^2 k}{2mg}\right] = \frac{{v_0}^2}{2g}$$