$$(3x^2y + 2xy + y^3)\,dx + (x^2 + y^2)\,dy = 0$$

$$\begin{cases} P_{(x,y)} = 3x^2y + 2xy + y^3 \;\rightarrow\; P_y = 3x^2 + 2x + 3y^2 \\ Q_{(x,y)} = x^2 + y^2 \;\rightarrow\; Q_x = 2x \end{cases} \Rightarrow\; P_y \neq Q_x \;\Rightarrow\; \textit{not exact}$$

$P\,dx + Q\,dy = 0$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

$P_y \quad Q_x$

$\rho = e^{\int F(x)\,dx}$, $F(x) = \frac{P_y - Q_x}{Q}$: $\rho = \rho(x)$

$\rho = e^{\int F(y)\,dy}$, $F(y) = \frac{Q_x - P_y}{P}$: $\rho = \rho(y)$

$P_1 = \rho \cdot P$

$Q_1 = \rho \cdot Q$

$$\Rightarrow \int_{x_0}^{x} P_1(x,y)\,dx + \int_{y_0}^{y} Q_1(x_0,y)\,dy = 0$$

$$F_{(x)} = \frac{P_y - Q_x}{Q} = \frac{3x^2 + 3y^2}{x^2 + y^2} = 3 \quad \Rightarrow \quad \rho_{(x)} = e^{\int 3dx} = e^{3x}$$

$$\begin{cases} P_{1(x,y)} = \rho P_{(x,y)} = e^{3x}(3x^2y + 2xy + y^3) \;\rightarrow\; P_{1y} = e^{3x}(3x^2 + 2x + 3y^2) \\ Q_{1(x,y)} = \rho Q_{(x,y)} = e^{3x}(x^2 + y^2) \;\rightarrow\; Q_{1x} = 3e^{3x}(x^2 + y^2) + 2xe^{3x} \end{cases} \Rightarrow P_{1y} = Q_{1x} \Rightarrow \textit{exact}$$

$$\int_{x_0}^{x} (3yx^2e^{3x} + 2yxe^{3x} + y^3e^{3x})\,dx + \int_{y_0}^{y} ({x_0}^2e^{3x_0} + e^{3x_0}y^2)dy = 0$$

$$\int x^2e^{3x}dx \;\rightarrow\; \begin{cases} u = x^2 \;\rightarrow\; du = 2xdx \\ dv = e^{3x}dx \;\rightarrow\; v = \frac{e^{3x}}{3} \end{cases} \qquad \int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int x^2 \cdot e^{3x}\,dx = \frac{x^2 \cdot e^{3x}}{3} - \frac{2}{3}\int xe^{3x}dx$$

$$\int xe^{3x}dx \;\rightarrow\; \begin{cases} u = x \;\rightarrow\; du = dx \\ dv = e^{3x}dx \;\rightarrow\; v = \frac{e^{3x}}{3} \end{cases} \qquad \int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int xe^{3x}\,dx = \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$$

$$\int x^2 \cdot e^{3x}\,dx = \frac{x^2 \cdot e^{3x}}{3} - \frac{2}{3}\left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right] = \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27}$$

$$\int_{x_0}^{x}(3yx^2e^{3x}+2yxe^{3x}+y^3e^{3x})\,dx+\int_{y_0}^{y}(x_0{}^2e^{3x_0}+e^{3x_0}y^2)dy=0$$

$$\left[yx^2e^{3x}-\frac{2yxe^{3x}}{3}+\frac{2ye^{3x}}{9}+\frac{2yxe^{3x}}{3}-\frac{2ye^{3x}}{9}+\frac{y^3e^{3x}}{3}\right|_{x_0}^{x}+\left[x_0{}^2e^{3x_0}y+\frac{e^{3x_0}y^3}{3}\right|_{y_0}^{y}=0$$

$$\left[yx^2e^{3x}+\frac{y^3e^{3x}}{3}\right|_{x_0}^{x}+\left[x_0{}^2e^{3x_0}y+\frac{e^{3x_0}y^3}{3}\right|_{y_0}^{y}=0$$

$$yx^2e^{3x}+\frac{y^3e^{3x}}{3}-yx_0{}^2e^{3x_0}-\frac{y^3e^{3x_0}}{3}+x_0{}^2e^{3x_0}y+\frac{e^{3x_0}y^3}{3}-x_0{}^2e^{3x_0}y_0-\frac{e^{3x_0}y_0{}^3}{3}=0$$

$$yx^2e^{3x}+\frac{y^3e^{3x}}{3}=C$$

$$3yx^2e^{3x}+y^3e^{3x}=C$$

$$e^{3x}(3yx^2+y^3)=C$$