נתון $w=f(u,v)$ ונתון $u=\frac{x^2-y^2}{2}, v=x\cdot y$

נתון גם כי w מקיימת את משוואת לפלס: $f_{uu}+f_{vv}=0$.

מצא את $w_{xx}+w_{yy}$

תשובה:

$$u=\frac{x^2-y^2}{2} \;=>\; \frac{\partial u}{\partial x}=x \;,\quad \frac{\partial u}{\partial y}=-y$$

$$, v=xy \;=>\; \frac{\partial v}{\partial x}=y \;,\quad \frac{\partial v}{\partial y}=x$$

$$w_x=\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}\cdot x+\frac{\partial f}{\partial v}\cdot y=f_u\cdot x+f_v\cdot y$$

$$w_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(x\cdot f_u+y\cdot f_v)=$$

$$=\frac{\partial}{\partial x}(x\cdot f_u)+\frac{\partial}{\partial x}(y\cdot f_v)=1\cdot f_u+x\cdot\frac{\partial}{\partial x}f_u+y\cdot\frac{\partial}{\partial x}f_v$$

$$w_y=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial f}{\partial u}\cdot(-y)+\frac{\partial f}{\partial v}\cdot x=f_u\cdot(-y)+f_v\cdot x$$

$$w_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(-y\cdot f_u+x\cdot f_v)=-\frac{\partial}{\partial y}(y\cdot f_u)+\frac{\partial}{\partial y}(x\cdot f_v)=$$

$$=-\left(1\cdot f_u+y\cdot\frac{\partial}{\partial y}f_u\right)+x\cdot\frac{\partial}{\partial y}f_v=-f_u-y\cdot\frac{\partial}{\partial y}f_u+x\cdot\frac{\partial}{\partial y}f_v$$

$$w_{xx}+w_{yy}=f_u+x\cdot\frac{\partial}{\partial x}f_u+y\cdot\frac{\partial}{\partial x}f_v-f_u-y\cdot\frac{\partial}{\partial y}f_u+x\cdot\frac{\partial}{\partial y}f_v=$$

$$=x\cdot\frac{\partial}{\partial x}f_u+y\cdot\frac{\partial}{\partial x}f_v-y\cdot\frac{\partial}{\partial y}f_u+x\cdot\frac{\partial}{\partial y}f_v=$$

$$=x\left(\frac{\partial}{\partial x}f_u+\frac{\partial}{\partial y}f_v\right)+y\left(\frac{\partial}{\partial x}f_v-\frac{\partial}{\partial y}f_u\right)=\cdots$$

$$\frac{\partial}{\partial x}f_u=\frac{\partial f_u}{\partial u}\cdot\frac{\partial u}{\partial x}=f_{uu}\cdot x \;,\quad \frac{\partial}{\partial x}f_v=\frac{\partial f_v}{\partial v}\cdot\frac{\partial v}{\partial x}=f_{vv}\cdot y$$

$$\frac{\partial}{\partial y}f_v=\frac{\partial f_v}{\partial v}\cdot\frac{\partial v}{\partial y}=f_{vv}\cdot x \;,\quad \frac{\partial}{\partial y}f_u=\frac{\partial f_u}{\partial u}\cdot\frac{\partial u}{\partial y}=f_{uu}\cdot(-y)=-f_{uu}\cdot y$$

$$\ldots=x(xf_{uu}+xf_{vv})+y(yf_{vv}+yf_{uu})=x^2(f_{uu}+f_{vv})+y^2(f_{vv}+f_{uu})=$$

$$=(x^2+y^2)(f_{vv}+f_{uu})=0$$